

# INEQUALITIES OF HERMITE-HADAMARD TYPE FOR EXTENDED s-CONVEX FUNCTIONS AND APPLICATIONS TO MEANS

BO-YAN XI AND FENG QI

ABSTRACT. In the paper, the authors introduce a new concept “extended  $s$ -convex functions”, establish some new integral inequalities of Hermite-Hadamard type for this kind of functions, and apply these inequalities to derive some inequalities of special means.

## 1. INTRODUCTION

Throughout this paper, we use the following notation:

$$(1.1) \quad \mathbb{R} = (-\infty, \infty), \quad \mathbb{R}_0 = [0, \infty), \quad \text{and} \quad \mathbb{R}_+ = (0, \infty).$$

The following definitions are well known in the literature.

**Definition 1.1.** A function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is said to be convex if

$$(1.2) \quad f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

holds for all  $x, y \in I$  and  $\lambda \in [0, 1]$ .

**Definition 1.2** ([5]). A function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}_0$  is said to be  $P$ -convex if

$$(1.3) \quad f(\lambda x + (1 - \lambda)y) \leq f(x) + f(y)$$

holds for all  $x, y \in I$  and  $\lambda \in [0, 1]$ .

**Definition 1.3** ([6]). A function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}_0$  is said to be a Godunova-Levin function if  $f$  is nonnegative and

$$(1.4) \quad f(\lambda x + (1 - \lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1 - \lambda}$$

holds for all  $x, y \in I$  and  $\lambda \in (0, 1)$ .

**Definition 1.4** ([7]). Let  $s \in (0, 1]$  be a real number. A function  $f : \mathbb{R}_0 \rightarrow \mathbb{R}_0$  is said to be  $s$ -convex (in the second sense) if

$$(1.5) \quad f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$

holds for all  $x, y \in I$  and  $\lambda \in [0, 1]$ .

In recent decades, a lot of inequalities of Hermite-Hadamard type for various kinds of convex functions have been established. Some of them may be recited as follows.

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**Theorem 1.1** ([4]). *Let  $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  and  $a, b \in I^\circ$  with  $a < b$ . If  $|f'(x)|$  is convex on  $[a, b]$ , then*

$$(1.6) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}.$$

**Theorem 1.2** ([9]). *Let  $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$  be differentiable on  $I^\circ$  and  $a, b \in I$  with  $a < b$ . If  $|f'(x)|^q$  is  $s$ -convex on  $[a, b]$  for some fixed  $s \in (0, 1]$  and  $q \geq 1$ , then*

$$(1.7) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \left( \frac{1}{2} \right)^{1-1/q} \left[ \frac{2 + 1/2^s}{(s+1)(s+2)} \right]^{1/q} [|f'(a)|^q + |f'(b)|^q]^{1/q}.$$

**Theorem 1.3** ([8]). *Let  $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$  be differentiable on  $I^\circ$ ,  $a, b \in I$  with  $a < b$ , and  $f' \in L[a, b]$ . If  $|f'(x)|^q$  is  $s$ -convex on  $[a, b]$  for some fixed  $s \in (0, 1]$  and  $q > 1$ , then*

$$(1.8) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left[ \frac{1}{(s+1)(s+2)} \right]^{1/q} \left( \frac{1}{2} \right)^{1/p} \times \left\{ \left[ |f'(a)|^q + (s+1) \left| f'\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + \left[ |f'(b)|^q + (s+1) \left| f'\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right\},$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Theorem 1.4** ([12]). *Let  $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$  be differentiable on  $I^\circ$ ,  $a, b \in I$  with  $a < b$ , and  $f' \in L[a, b]$ . If  $|f'(x)|$  is  $s$ -convex on  $[a, b]$  for some  $s \in (0, 1]$ , then*

$$(1.9) \quad \left| \frac{1}{6} \left[ f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(s-4)6^{s+1} + 2 \times 5^{s+2} - 2 \times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)} (b-a)(|f'(a)| + |f'(b)|),$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

Some inequalities of Hermite-Hadamard type were also obtained in [1, 2, 3, 10, 11, 13, 14, 15, 16, 17, 18] and related references therein.

In this paper, we will introduce a new concept “extended  $s$ -convex functions”, establish some new integral inequalities of Hermite-Hadamard type for extended  $s$ -convex functions, and apply these newly established integral inequalities to derive some inequalities of special means. These results generalize inequalities stated in Theorems 1.1 to 1.4.

## 2. DEFINITION AND LEMMAS

We first define the concept “extended  $s$ -convex functions” and establish an integral identity.

**Definition 2.1.** For some  $s \in [-1, 1]$ , a function  $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}_0$  is said to be extended  $s$ -convex if

$$(2.1) \quad f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all  $x, y \in I$  and  $\lambda \in (0, 1)$ .

It is obvious that the extended 1-convex function, 0-convex function, and  $-1$ -convex function are just the usually convex function in Definition 1.1, the  $P$ -convex functions in Definition 1.2, and Godunova-Levin convex function in Definition 1.3, respectively. It is also clear that Definition 2.1 extends Definition 1.4.

For establishing new integral inequalities of Hermite-Hadamard type for extended  $s$ -convex functions, we need the following integral identity.

**Lemma 2.1.** *Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be differentiable on  $I^\circ$  and  $a, b \in I$  with  $a < b$ . If  $f' \in L[a, b]$  and  $\lambda, \mu \in \mathbb{R}$ , then*

$$\begin{aligned} & \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{b-a}{4} \int_0^1 \left[ (1-\lambda-t)f'\left(ta + (1-t)\frac{a+b}{2}\right) + (\mu-t)f'\left(t\frac{a+b}{2} + (1-t)b\right) \right] dt. \end{aligned}$$

*Proof.* integrating by parts and changing variable of definite integral yield

$$\begin{aligned} & \int_0^1 (1-\lambda-t)f'\left(ta + (1-t)\frac{a+b}{2}\right) dt \\ &= -\frac{2}{b-a} \left[ (1-\lambda-t)f\left(ta + (1-t)\frac{a+b}{2}\right) \Big|_0^1 + \int_0^1 f\left(ta + (1-t)\frac{a+b}{2}\right) dt \right] \\ &= \frac{2}{b-a} \left[ \lambda f(a) + (1-\lambda)f\left(\frac{a+b}{2}\right) \right] - \frac{4}{(b-a)^2} \int_a^{(a+b)/2} f(x) dx \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 (\mu-t)f'\left(t\frac{a+b}{2} + (1-t)b\right) dt \\ &= -\frac{2}{b-a} \left[ (\mu-t)f\left(t\frac{a+b}{2} + (1-t)b\right) \Big|_0^1 + \int_0^1 f\left(t\frac{a+b}{2} + (1-t)b\right) dt \right] \\ &= \frac{2}{b-a} \left[ (1-\mu)f\left(\frac{a+b}{2}\right) + \mu f(b) \right] - \frac{4}{(b-a)^2} \int_{(a+b)/2}^b f(x) dx. \end{aligned}$$

Adding these two equations leads to Lemma 2.1. □

**Lemma 2.2.** *Let  $s > -1$ ,  $0 \leq \xi \leq 1$ ,  $\omega \in \mathbb{R} \setminus \{0\}$ ,  $\eta \geq 0$ , and  $\omega + \eta \geq 0$ . Then*

$$\begin{aligned} (2.2) \quad & \int_0^1 |\xi - t|(\omega t + \eta)^s dt \\ &= \frac{2(\omega\xi + \eta)^{s+2} - [\eta + (s+2)\omega\xi]\eta^{s+1} - [2\omega\xi + \eta + s\omega(\xi - 1) - \omega](\omega + \eta)^{s+1}}{\omega^2(s+1)(s+2)}. \end{aligned}$$

*In particular, if  $(\omega, \eta) = (1, 0)$ ,  $(1, 1)$ ,  $(-1, 1)$ , or  $(-1, 2)$  respectively, then*

$$\begin{aligned} & \int_0^1 |\xi - t|t^s dt = \frac{2\xi^{s+2} - (s+2)\xi + s+1}{(s+1)(s+2)}, \\ & \int_0^1 |\xi - t|(1+t)^s dt = \frac{2(\xi+1)^{s+2} - [(s+2)\xi - s]2^{s+1} - (s+2)\xi - 1}{(s+1)(s+2)}, \end{aligned}$$

$$\begin{aligned}\int_0^1 |\xi - t|(1-t)^s dt &= \frac{2(1-\xi)^{s+2} + (s+2)\xi - 1}{(s+1)(s+2)}, \\ \int_0^1 |\xi - t|(2-t)^s dt &= \frac{2(2-\xi)^{s+2} + [(s+2)\xi - 2]2^{s+1} + (s+2)\eta - s - 3}{(s+1)(s+2)}.\end{aligned}$$

*Proof.* These follow from straightforward computation of definite integrals.  $\square$

### 3. SOME INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE

Now we are in a position to establish some new integral inequalities of Hermite-Hadamard type for differentiable extended  $s$ -convex functions.

**Theorem 3.1.** *Let  $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$  be differentiable on  $I^\circ$ ,  $a, b \in I$  with  $a < b$ ,  $f' \in L[a, b]$ , and  $0 \leq \lambda, \mu \leq 1$ . If  $|f'(x)|^q$  for  $q \geq 1$  is an extended  $s$ -convex function on  $[a, b]$  for some fixed  $s \in [-1, 1]$ , then*

(1) *when  $-1 < s \leq 1$ , we have*

$$\begin{aligned}& \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2^{s/q+2}} \left[ \frac{1}{(s+1)(s+2)} \right]^{1/q} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} [ \{ 2(2-\lambda)^{s+2} \right. \right. \\ & \quad \left. \left. + [(s+2)\lambda - 2]2^{s+1} + (s+2)\lambda - s - 3 \} |f'(a)|^q + \{ 2\lambda^{s+2} - (s+2)\lambda + s \right. \right. \\ & \quad \left. \left. + 1 \} |f'(b)|^q \right]^{1/q} + \left( \frac{1}{2} - \mu + \mu^2 \right)^{1-1/q} \left[ (2\mu^{s+2} - (s+2)\mu + s + 1) |f'(a)|^q \right. \right. \\ & \quad \left. \left. + \{ 2(2-\mu)^{s+2} + [(s+2)\mu - 2]2^{s+1} + (s+2)\mu - s - 3 \} |f'(b)|^q \right]^{1/q} \right\};\end{aligned}$$

(2) *when  $s = -1$ , we have*

$$\begin{aligned}(3.1) \quad & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2^{3-2/q}} \left\{ [(2\ln 2 - 1)|f'(a)|^q + |f'(b)|^q]^{1/q} + [|f'(a)|^q + (2\ln 2 - 1)|f'(b)|^q]^{1/q} \right\}.\end{aligned}$$

*Proof.* For  $-1 < s \leq 1$ , since  $|f'(x)|^q$  is extended  $s$ -convex on  $[a, b]$ , by Lemmas 2.1 and 2.2 and by Hölder integral inequality, we have

$$\begin{aligned}& \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} \left[ \int_0^1 |1-\lambda-t| \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) \right| dt + \int_0^1 |\mu-t| \left| f'\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \\ & = \frac{b-a}{4} \left[ \int_0^1 |1-\lambda-t| \left| f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) \right| dt + \int_0^1 |\mu-t| \left| f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \right| dt \right] \\ & \leq \frac{b-a}{2^{s/q+2}} \left\{ \left( \int_0^1 |1-\lambda-t| dt \right)^{1-1/q} \left[ \int_0^1 |1-\lambda-t| ((1+t)^s |f'(a)|^q + (1-t)^s |f'(b)|^q) dt \right]^{1/q} \right. \\ & \quad \left. + \left( \int_0^1 |\mu-t| dt \right)^{1-1/q} \left[ \int_0^1 |\mu-t| ((1+t)^s |f'(a)|^q + (1-t)^s |f'(b)|^q) dt \right]^{1/q} \right\}.\end{aligned}$$

$$\begin{aligned}
& + \left( \int_0^1 |\mu - t| dt \right)^{1-1/q} \left[ \int_0^1 |\mu - t| (t^s |f'(a)|^q + (2-t)^s |f'(b)|^q) dt \right]^{1/q} \Big\} \\
& = \frac{b-a}{2^{s/q+2}} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} \left[ \frac{1}{(s+1)(s+2)} [(2(2-\lambda)^{s+2} + ((s+2)\lambda - 2)2^{s+1} \right. \right. \\
& \quad \left. \left. + (s+2)\lambda - s - 3) |f'(a)|^q + (2\lambda^{s+2} + s + 1 - (s+2)\lambda) |f'(b)|^q] \right]^{1/q} \right. \\
& \quad \left. + \left( \frac{1}{2} - \mu + \mu^2 \right)^{1-1/q} \left[ \frac{1}{(s+1)(s+2)} [(2\mu^{s+2} + s + 1 - (s+2)\mu) |f'(a)|^q \right. \right. \\
& \quad \left. \left. + (2(2-\mu)^{s+2} + ((s+2)\mu - 2)2^{s+1} + (s+2)\mu - s - 3) |f'(a)|^q] \right]^{1/q} \right\}.
\end{aligned}$$

For  $s = -1$ , since  $|f'(x)|^q$  is extended  $-1$ -convex on  $[a, b]$ , by Lemma 2.1 and Hölder integral inequality, we have

$$\begin{aligned}
& \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{b-a}{4} \left[ \int_0^1 \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) \right| (1-t) dt + \int_0^1 t \left| f'\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \\
& \leq \frac{b-a}{2^{2-1/q}} \left\{ \left[ \int_0^1 (1-t) dt \right]^{1-1/q} \left[ \int_0^1 (1-t) ((1+t)^{-1} |f'(a)|^q + (1-t)^{-1} |f'(b)|^q) dt \right]^{1/q} \right. \\
& \quad \left. + \left( \int_0^1 t dt \right)^{1-1/q} \left[ \int_0^1 t (t^{-1} |f'(a)|^q + (2-t)^{-1} |f'(b)|^q) dt \right]^{1/q} \right\} \\
& = \frac{b-a}{2^{3-2/q}} \left\{ [(2 \ln 2 - 1) |f'(a)|^q + |f'(b)|^q]^{1/q} + [|f'(a)|^q + (2 \ln 2 - 1) |f'(b)|^q]^{1/q} \right\}.
\end{aligned}$$

Theorem 3.1 is proved.  $\square$

**Corollary 3.1.1.** *Under conditions of Theorem 3.1,*

(1) *if  $q = 1$  and  $-1 < s \leq 1$ , we have*

$$\begin{aligned}
(3.2) \quad & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{b-a}{2^{s+2}(s+1)(s+2)} \left\{ [2(2-\lambda)^{s+2} + 2\mu^{s+2} + ((s+2)\lambda - 2)2^{s+1} \right. \\
& \quad \left. + (s+2)(\lambda - \mu) - 2] |f'(a)| + [2\lambda^{s+2} + 2(2-\mu)^{s+2} \right. \\
& \quad \left. + ((s+2)\mu - 2)2^{s+1} + (s+2)(\mu - \lambda) - 2] |f'(b)| \right\};
\end{aligned}$$

(2) *if  $q = 1$  and  $s = -1$ , we have*

$$(3.3) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a)(\ln 2) [|f'(a)| + |f'(b)|].$$

**Corollary 3.1.2.** *Under conditions of Theorem 3.1,*

(1) when  $\lambda = \mu$  and  $-1 < s \leq 1$ , we have

$$\begin{aligned} & \left| \lambda \frac{f(a) + f(b)}{2} + (1 - \lambda) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2^{s/q+2}} \left[ \frac{1}{(s+1)(s+2)} \right]^{1/q} \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} \{ [(2(2-\lambda)^{s+2} \\ & + ((s+2)\lambda - 2)2^{s+1} + (s+2)\lambda - s - 3]|f'(a)|^q + (2\lambda^{s+2} + s + 1 \\ & - (s+2)\lambda)|f'(b)|^q]^{1/q} + [(2\lambda^{s+2} - (s+2)\lambda + s + 1)|f'(a)|^q \\ & + (2(2-\lambda)^{s+2} + ((s+2)\lambda - 2)2^{s+1} + (s+2)\lambda - s - 3)|f'(b)|^q]^{1/q} \}; \end{aligned}$$

(2) when  $\lambda = \mu$ ,  $-1 < s \leq 1$ , and  $q = 1$ ,

$$\begin{aligned} (3.4) \quad & \left| \lambda \frac{f(a) + f(b)}{2} + (1 - \lambda) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)\{(2-\lambda)^{s+2} + \lambda^{s+2} + [(s+2)\lambda - 2]2^s - 1\}(|f'(a)| + |f'(b)|)}{(s+1)(s+2)2^{s+1}}; \end{aligned}$$

(3) when  $\lambda = \mu$ ,  $-1 < s \leq 1$ , and  $\lambda = \mu = 1$ , we have

$$\begin{aligned} (3.5) \quad & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{8} \left[ \frac{2}{(s+1)(s+2)} \right]^{1/q} \left\{ \left[ \left( 2s + \frac{1}{2^s} \right) |f'(a)|^q + \frac{|f'(b)|^q}{2^s} \right]^{1/q} \right. \\ & \quad \left. + \left[ \left( 2s + \frac{1}{2^s} \right) |f'(b)|^q + \frac{|f'(a)|^q}{2^s} \right]^{1/q} \right\} \\ & \leq \frac{(b-a)[|f'(a)|^q + |f'(b)|^q]^{1/q}}{4} \left[ \frac{4 + (1/2)^{s-1}}{(s+1)(s+2)} \right]^{1/q}. \end{aligned}$$

*Remark 3.1.* The inequality (1.7) is a special case of (3.5) applied to  $0 < s \leq 1$ . The inequality (1.9) can be deduced from (3.4) applied to  $\lambda = \mu = \frac{1}{3}$  and  $0 < s \leq 1$ . These show that Theorem 3.1 and its corollaries generalize some main results obtained in [9, 12].

**Corollary 3.1.3.** *Under conditions of Theorem 3.1,*

(1) when  $s = 1$ , we have

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2^{1/q+2}} \left( \frac{1}{6} \right)^{1/q} \\ & \quad \times \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} [(4 - 9\lambda + 12\lambda^2 - 2\lambda^3)|f'(a)|^q + (2 - 3\lambda + 2\lambda^3)|f'(b)|^q]^{1/q} \right. \\ & \quad \left. + \left( \frac{1}{2} - \mu + \mu^2 \right)^{1-1/q} [(2 - 3\mu + 2\mu^3)|f'(a)|^q + (4 - 9\mu + 12\mu^2 - 2\mu^3)|f'(b)|^q]^{1/q} \right\}; \end{aligned}$$

(2) when  $s = 1$  and  $q = 1$ , we have

$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{b-a}{48} \{ (6 - 9\lambda + 12\lambda^2 - 2\lambda^3 - 3\mu + 2\mu^3) |f'(a)| + (6 + 3\lambda + 2\lambda^3 - 9\mu + 12\mu^2 - 2\mu^3) |f'(b)| \};$$

(3) when  $s = 1$  and  $\lambda = \mu$ ,

$$\left| \frac{\lambda[f(a) + f(b)]}{2} + (1 - \lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{1}{12}\right)^{1/q} \left(\frac{1}{2} - \lambda + \lambda^2\right)^{1-1/q}$$

$$\times \{ [(4 - 9\lambda + 12\lambda^2 - 2\lambda^3) |f'(a)|^q + (2 - 3\lambda + 2\lambda^3) |f'(b)|^q]^{1/q}$$

$$+ [(2 - 3\lambda + 2\lambda^3) |f'(a)|^q + (4 - 9\lambda + 12\lambda^2 - 2\lambda^3) |f'(b)|^q]^{1/q} \};$$

(4) when  $s = 1$ ,  $q = 1$ , and  $\lambda = \mu$ , we have

$$(3.6) \quad \left| \frac{\lambda[f(a) + f(b)]}{2} + (1 - \lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{b-a}{8} (1 - 2\lambda + 2\lambda^2) [|f'(a)| + |f'(b)|].$$

*Remark 3.2.* Letting  $\lambda = 1$  in (3.6) yields the inequality (1.6) in [4].

**Corollary 3.1.4.** Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be differentiable on  $I^\circ$ ,  $a, b \in I$  with  $a < b$ , and  $f' \in L[a, b]$ . If  $|f'(x)|^q$  is convex on  $[a, b]$  for  $q \geq 1$ , then

$$\left| \frac{1}{2} \left[ \frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{b-a}{16} \left\{ \left[ \frac{3|f'(a)|^q + |f'(b)|^q}{4} \right]^{1/q} + \left[ \frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right]^{1/q} \right\},$$

$$\left| \frac{1}{3} \left[ f(a) + f(b) + f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{5(b-a)}{72} \left\{ \left[ \frac{37|f'(a)|^q + 8|f'(b)|^q}{45} \right]^{1/q} + \left[ \frac{8|f'(a)|^q + 37|f'(b)|^q}{45} \right]^{1/q} \right\},$$

$$\left| \frac{1}{6} \left[ f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{5(b-a)}{72} \left\{ \left[ \frac{61|f'(a)|^q + 29|f'(b)|^q}{90} \right]^{1/q} + \left[ \frac{29|f'(a)|^q + 61|f'(b)|^q}{90} \right]^{1/q} \right\}.$$

**Theorem 3.2.** Let  $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$  be differentiable on  $I^\circ$ ,  $a, b \in I$  with  $a < b$ , and  $f' \in L[a, b]$ . If  $|f'(x)|^q$  for  $q \geq 1$  is an extended  $s$ -convex function on  $[a, b]$ , then for  $s \in (-1, 1]$  and  $0 \leq \lambda, \mu \leq 1$ ,

$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{b-a}{4} \left[ \frac{1}{(s+1)(s+2)} \right]^{1/q} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} \left[ (2(1-\lambda)^{s+2} + (s+2)\lambda - 1) |f'(a)|^q \right. \right.$$

$$\left. \left. + (2\lambda^{s+2} + s + 1 - (s+2)\lambda) \left| f'\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + \left( \frac{1}{2} - \mu + \mu^2 \right)^{1-1/q} \left[ (2\mu^{s+2} + s + 1 \right. \right.$$

$$\begin{aligned}
& - (s+2)\mu \left| f' \left( \frac{a+b}{2} \right) \right|^q + (2(1-\mu)^{s+2} + (s+2)\mu - 1) |f'(b)|^q \Big]^{1/q} \Big\} \\
& \leq \frac{b-a}{2^{s/q+2}} \left[ \frac{1}{(s+1)(s+2)} \right]^{1/q} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} \left[ ((1-\lambda)^{s+2} 2^{s+1} + 2\lambda^{s+2} \right. \right. \\
& \quad \left. \left. + s+1 + ((s+2)\lambda - 1)2^s - (s+2)\lambda \right) |f'(a)|^q + (2\lambda^{s+2} - (s+2)\lambda + s+1) |f'(b)|^q \right]^{1/q} \\
& \quad + \left( \frac{1}{2} - \mu + \mu^2 \right)^{1-1/q} \left[ (2\mu^{s+2} - (s+2)\mu + s+1) |f'(a)|^q \right. \\
& \quad \left. \left. + ((1-\mu)^{s+2} 2^{s+1} + 2\mu^{s+1} + ((s+2)\mu - 1)2^s - (s+2)\mu + s+1) |f'(b)|^q \right]^{1/q} \right\}.
\end{aligned}$$

*Proof.* By similar arguments as in the proof of Theorem 3.1 and by the extended  $s$ -convexity of the function  $|f'(x)|^q$ , we have

$$\begin{aligned}
& \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{b-a}{4} \left\{ \left( \int_0^1 |1-\lambda-t| dt \right)^{1-1/q} \left[ \int_0^1 |1-\lambda-t| \left( (1-t)^s \left| f' \left( \frac{a+b}{2} \right) \right|^q + t^s |f'(a)|^q \right) dt \right]^{1/q} \right. \\
& \quad \left. + \left( \int_0^1 |\mu-t| dt \right)^{1-1/q} \left[ \int_0^1 |\mu-t| \left( t^s \left| f' \left( \frac{a+b}{2} \right) \right|^q + (1-t)^s |f'(b)|^q \right) dt \right]^{1/q} \right\} \\
& = \frac{b-a}{4} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} \left[ \frac{1}{(s+1)(s+2)} \left( (2(1-\lambda)^{s+2} + (s+2)\lambda - 1) |f'(a)|^q \right. \right. \right. \\
& \quad \left. \left. + (2\lambda^{s+2} - (s+2)\lambda + s+1) \left| f' \left( \frac{a+b}{2} \right) \right|^q \right) \right]^{1/q} + \left( \frac{1}{2} - \mu + \mu^2 \right)^{1-1/q} \left[ \frac{1}{(s+1)(s+2)} \right. \right. \\
& \quad \left. \left. \times \left( (2\mu^{s+2} + s+1 - (s+2)\mu) \left| f' \left( \frac{a+b}{2} \right) \right|^q + (2(1-\mu)^{s+2} + (s+2)\mu - 1) |f'(a)|^q \right) \right]^{1/q} \right\}.
\end{aligned}$$

Combining this with

$$\left| f' \left( \frac{a+b}{2} \right) \right|^q \leq \left( \frac{1}{2} \right)^s [|f'(a)|^q + |f'(b)|^q]$$

leads to Theorem 3.2. □

**Corollary 3.2.1.** *Under conditions of Theorem 3.2, when  $q = 1$ , we have*

$$\begin{aligned}
& \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{b-a}{4(s+1)(s+2)} \left\{ [2(1-\lambda)^{s+2} + (s+2)\lambda - 1] |f'(a)| + [2\lambda^{s+2} + 2\mu^{s+2} \right. \\
& \quad \left. + (s+2)(1-\lambda-\mu) + s] \left| f' \left( \frac{a+b}{2} \right) \right| + [2(1-\mu)^{s+2} + (s+2)\mu - 1] |f'(b)| \right\} \\
& \leq \frac{b-a}{2^{s+2}(s+1)(s+2)} \left\{ [(1-\lambda)^{s+2} 2^{s+1} + 2\lambda^{s+2} + 2\mu^{s+2} + ((s+2)\lambda - 1)2^s \right.
\end{aligned}$$



$$+ (s+2)(1-\lambda-\mu) + s] |f'(a)| + [2\lambda^{s+2} + (1-\mu)^{s+2} 2^{s+1} + 2\mu^{s+2} + (s+2)(1-\lambda-\mu) + ((s+2)\mu - 1)2^s + s] |f'(b)| \}.$$

**Corollary 3.2.2.** *Under conditions of Theorem 3.2, if  $\lambda = \mu$ , then*

$$\begin{aligned} & \left| \frac{\lambda[f(a) + f(b)]}{2} + (1-\lambda)2f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} \left[ \frac{1}{(s+1)(s+2)} \right]^{1/q} \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} \left\{ \left[ (2(1-\lambda)^{s+2} + (s+2)\lambda - 1) |f'(a)|^q \right. \right. \\ & \quad + (2\lambda^{s+2} + s+1 - (s+2)\lambda) \left| f'\left(\frac{a+b}{2}\right) \right|^q \Big]^{1/q} + \left[ (2\mu^{s+2} + s+1 \right. \\ & \quad \left. - (s+2)\lambda) \left| f'\left(\frac{a+b}{2}\right) \right|^q + (2(1-\lambda)^{s+2} + (s+2)\lambda - 1) |f'(b)|^q \right]^{1/q} \Big\} \\ & \leq \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} \frac{b-a}{2^{s/q+2}} \left[ \frac{1}{(s+1)(s+2)} \right]^{1/q} \left\{ [(2^{s+1}(1-\lambda)^{s+2} + 2\lambda^{s+2} - (s+2)\lambda \right. \\ & \quad + ((s+2)\lambda - 1)2^s + s+1) |f'(a)|^q + (2\lambda^{s+2} - (s+2)\lambda + s+1) |f'(b)|^q]^{1/q} \\ & \quad + [(2\lambda^{s+2} - (s+2)\lambda + s+1) |f'(a)|^q + (2\lambda^{s+1} + (1-\lambda)^{s+2} 2^{s+1} + ((s+2)\lambda - 1)2^s \\ & \quad \left. - (s+2)\lambda + s+1) |f'(b)|^q]^{1/q} \right\}. \end{aligned}$$

*Remark 3.3.* The inequality (1.8) can be deduced from letting  $\lambda = \mu = 0$  in Corollary 3.2.2.

**Corollary 3.2.3.** *Under conditions of Theorem 3.2, when  $s = 1$ , we have*

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} \left( \frac{1}{6} \right)^{1/q} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} \left[ (1-3\lambda+6\lambda^2-2\lambda^3) |f'(a)|^q \right. \right. \\ & \quad + (2\lambda^3-3\lambda+3) \left| f'\left(\frac{a+b}{2}\right) \right|^q \Big]^{1/q} + \left( \frac{1}{2} - \mu + \mu^2 \right)^{1-1/q} \left[ (2\mu^3-3\mu+2) \left| f'\left(\frac{a+b}{2}\right) \right|^q \right. \\ & \quad \left. \left. + (1-3\mu+6\mu^2-2\mu^3) |f'(b)|^q \right]^{1/q} \right\} \\ & \leq \frac{b-a}{2^{1/q+2}} \left( \frac{1}{6} \right)^{1/q} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} [(4-9\lambda+12\lambda^2-2\lambda^3) |f'(a)|^q \right. \\ & \quad + (2\lambda^3-3\lambda+2) |f'(b)|^q]^{1/q} + \left( \frac{1}{2} - \mu + \mu^2 \right)^{1-1/q} [(2\mu^3-3\mu+2) |f'(a)|^q \\ & \quad \left. + (4-9\mu+12\mu^2-2\mu^3) |f'(b)|^q]^{1/q} \right\}. \end{aligned}$$

**Theorem 3.3.** *Let  $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$  be differentiable on  $I^\circ$ ,  $a, b \in I$  with  $a < b$ , and  $f' \in L[a, b]$ . If  $|f'(x)|^q$  for  $q \geq 1$  is an extended  $s$ -convex function on  $[a, b]$ , then for  $s \in (-1, 1]$  and  $0 \leq \lambda, \mu \leq 1$ ,*

(1) *when  $q = 1$ , we have*

$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2^{s+2}(s+1)} \\ \times \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right) [|f'(a)| + (2^{s+1} - 1)|f'(b)|] + \left( \frac{1}{2} - \mu + \mu^2 \right) [(2^{s+1} - 1)|f'(a)| + |f'(b)|] \right\};$$

(2) when  $q > 1$ , we have

$$(3.7) \quad \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2^{s/q+2}} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \\ \times \left( \frac{1}{s+1} \right)^{1/q} \left\{ [(1-\lambda)^{(2q-1)/(q-1)} + \lambda^{(2q-1)/(q-1)}]^{1-1/q} [(2^{s+1} - 1)|f'(a)|^q + |f'(b)|^q]^{1/q} \right. \\ \left. + [\mu^{(2q-1)/(q-1)} + (1-\mu)^{(2q-1)/(q-1)}]^{1-1/q} [|f'(a)|^q + (2^{s+1} - 1)|f'(b)|^q]^{1/q} \right\}.$$

*Proof.* For  $q > 1$ , by the extended  $s$ -convexity of  $|f'(x)|^q$  on  $[a, b]$ , Lemma 2.1, and Hölder integral inequality, we have

$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{b-a}{4} \left[ \int_0^1 |1 - \lambda - t| \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) \right| dt + \int_0^1 |\mu - t| \left| f'\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \\ \leq \frac{b-a}{2^{s/q+2}} \left\{ \left( \int_0^1 |1 - \lambda - t|^{q/(q-1)} dt \right)^{1-1/q} \left[ \int_0^1 ((1+t)^s |f'(a)|^q + (1-t)^s |f'(b)|^q) dt \right]^{1/q} \right. \\ \left. + \left( \int_0^1 |\mu - t|^{q/(q-1)} dt \right)^{1-1/q} \left[ \int_0^1 (t^s |f'(a)|^q + (2-t)^s |f'(b)|^q) dt \right]^{1/q} \right\}.$$

In virtue of Lemma 2.2, a direct calculation yields

$$\int_0^1 |1 - \lambda - t|^{q/(q-1)} dt = \frac{q-1}{2q-1} [\lambda^{(2q-1)/(q-1)} + (1-\lambda)^{(2q-1)/(q-1)}], \\ \int_0^1 |\mu - t|^{q/(q-1)} dt = \frac{q-1}{2q-1} [\mu^{(2q-1)/(q-1)} + (1-\mu)^{(2q-1)/(q-1)}].$$

A straightforward computation gives

$$\int_0^1 [(1+t)^s |f'(a)|^q + (1-t)^s |f'(b)|^q] dt = \frac{(2^{s+1} - 1)|f'(a)|^q + |f'(b)|^q}{s+1}, \\ \int_0^1 [t^s |f'(a)|^q + (2-t)^s |f'(b)|^q] dt = \frac{|f'(a)|^q + (2^{s+1} - 1)|f'(b)|^q}{s+1}.$$

Substituting the last four equalities into the first inequality and simplifying establish the inequality (3.7).

For  $q = 1$ , utilizing the extended  $s$ -convexity of  $|f'(x)|^q$  on  $[a, b]$ , Lemma 2.1, and Hölder integral inequality, we have

$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{b-a}{4} \left[ \int_0^1 |1 - \lambda - t| \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) \right| dt + \int_0^1 |\mu - t| \left| f'\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right]$$

$$\begin{aligned}
&\leq \frac{b-a}{2^{s+2}} \left\{ \left( \int_0^1 |1-\lambda-t| dt \right) \int_0^1 [(1+t)^s |f'(a)| + (1-t)^s |f'(b)|] dt \right. \\
&\quad \left. + \left( \int_0^1 |\mu-t| dt \right) \int_0^1 [t^s |f'(a)| + (2-t)^s |f'(b)|] dt \right\} \\
&= \frac{b-a}{2^{s+2}(s+1)} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right) [|f'(a)| + (2^{s+1}-1)|f'(b)|] \right. \\
&\quad \left. + \left( \frac{1}{2} - \mu + \mu^2 \right) [(2^{s+1}-1)|f'(a)| + |f'(b)|] \right\}.
\end{aligned}$$

Theorem 3.3 is thus proved.  $\square$

**Corollary 3.3.1.** *Under conditions of Theorem 3.3,*

(1) *when  $\lambda = \mu$  and  $q > 1$ , we have*

$$\begin{aligned}
&\left| \frac{\lambda[f(a) + f(b)]}{2} + (1-\lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
&\leq \frac{b-a}{2^{s/q+2}} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left( \frac{1}{s+1} \right)^{1/q} [\lambda^{(2q-1)/(q-1)} + (1-\lambda)^{(2q-1)/(q-1)}]^{1-1/q} \\
&\quad \times \{ [(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q]^{1/q} + [|f'(a)|^q + (2^{s+1}-1)|f'(b)|^q]^{1/q} \};
\end{aligned}$$

(2) *when  $\lambda = \mu = 0, 1$  and  $q > 1$ , we have*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2^{s/q+2}} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left( \frac{1}{s+1} \right)^{1/q} \\
&\quad \times \{ [(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q]^{1/q} + [|f'(a)|^q + (2^{s+1}-1)|f'(b)|^q]^{1/q} \}
\end{aligned}$$

and

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2^{s/q+2}} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left( \frac{1}{s+1} \right)^{1/q} \\
&\quad \times \{ [(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q]^{1/q} + [|f'(a)|^q + (2^{s+1}-1)|f'(b)|^q]^{1/q} \}.
\end{aligned}$$

**Corollary 3.3.2.** *Under conditions of Theorem 3.3,*

(1) *when  $q = 1$  and  $s = 1$ , we have*

$$\begin{aligned}
&\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
&\leq \frac{b-a}{16} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right) [|f'(a)| + 3|f'(b)|] + \left( \frac{1}{2} - \mu + \mu^2 \right) [3|f'(a)| + |f'(b)|] \right\};
\end{aligned}$$

(2) *when  $q > 1$  and  $s = 1$ , we have*

$$\begin{aligned}
&\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2^{2/q+2}} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \\
&\quad \times \{ [\lambda^{(2q-1)/(q-1)} + (1-\lambda)^{(2q-1)/(q-1)}]^{1-1/q} [3|f'(a)|^q + |f'(b)|^q]^{1/q} \\
&\quad + [\mu^{(2q-1)/(q-1)} + (1-\mu)^{(2q-1)/(q-1)}]^{1-1/q} [|f'(a)|^q + 3|f'(b)|^q]^{1/q} \};
\end{aligned}$$

(3) when  $q = 1$ ,  $\lambda = \mu$ , and  $s = 1$ , we have

$$(3.8) \quad \left| \frac{\lambda[f(a) + f(b)]}{2} + (1 - \lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \{ (1 - 2\lambda + 2\lambda^2) [|f'(a)| + |f'(b)|] \};$$

(4) when  $q > 1$ ,  $\lambda = \mu$ , and  $s = 1$ , we have

$$(3.9) \quad \left| \frac{\lambda[f(a) + f(b)]}{2} + (1 - \lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \left( \frac{q-1}{2q-1} \right)^{1-1/q} \frac{b-a}{2^{2/q}} \times [\lambda^{(2q-1)/(q-1)} + (1 - \lambda)^{(2q-1)/(q-1)}]^{1-1/q} [|f'(a)|^q + |f'(b)|^q]^{1/q}.$$

**Theorem 3.4.** Let  $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$  be differentiable on  $I^\circ$ ,  $a, b \in I$  with  $a < b$ , and  $f' \in L[a, b]$ . If  $|f'(x)|^q$  for  $q \geq 1$  is an extended  $s$ -convex function on  $[a, b]$ , then, for  $s \in (-1, 1]$  and  $0 \leq \lambda, \mu \leq 1$ ,

(1) when  $q = 1$ , we have

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4(s+1)} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right) \left[ |f'(a)| + \left| f'\left(\frac{a+b}{2}\right) \right| \right] \right. \\ & \quad \left. + \left( \frac{1}{2} - \mu + \mu^2 \right) \left[ \left| f'\left(\frac{a+b}{2}\right) \right| + |f'(b)| \right] \right\} \\ & \leq \frac{b-a}{2^{s+2}(s+1)} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right) [(2^s + 1)|f'(a)| + |f'(b)|] \right. \\ & \quad \left. + \left( \frac{1}{2} - \mu + \mu^2 \right) [|f'(a)| + (2^s + 1)|f'(b)|] \right\}; \end{aligned}$$

(2) when  $q > 1$ , we have

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left( \frac{1}{s+1} \right)^{1/q} \left\{ [\lambda^{(2q-1)/(q-1)} + (1 - \lambda)^{(2q-1)/(q-1)}]^{1-1/q} \right. \\ & \quad \times \left[ |f'(a)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + [\mu^{(2q-1)/(q-1)} + (1 - \mu)^{(2q-1)/(q-1)}]^{1-1/q} \\ & \quad \times \left[ \left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(b)|^q \right]^{1/q} \left. \right\} \\ & \leq \frac{b-a}{2^{s/q+2}} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left( \frac{1}{s+1} \right)^{1/q} \\ & \quad \times \left\{ [\lambda^{(2q-1)/(q-1)} + (1 - \lambda)^{(2q-1)/(q-1)}]^{1-1/q} [(2^s + 1)|f'(a)|^q + |f'(b)|^q]^{1/q} \right. \\ & \quad \left. + [\mu^{(2q-1)/(q-1)} + (1 - \mu)^{(2q-1)/(q-1)}]^{1-1/q} [|f'(a)|^q + (2^s + 1)|f'(b)|^q]^{1/q} \right\}. \end{aligned}$$

*Proof.* For  $q > 1$ , since  $|f'(x)|^q$  is extended  $s$ -convex on  $[a, b]$ , by Lemma 2.1 and Hölder integral inequality, we have

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} \left[ \int_0^1 |1 - \lambda - t| \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) \right| dt + \int_0^1 |\mu - t| \left| f'\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \\ & \leq \frac{b-a}{4} \left\{ \left( \int_0^1 |1 - \lambda - t|^{q/(q-1)} dt \right)^{1-1/q} \left[ \int_0^1 \left( t^s |f'(a)|^q + (1-t)^s \left| f'\left(\frac{a+b}{2}\right) \right|^q \right) dt \right]^{1/q} \right. \\ & \quad \left. + \left( \int_0^1 |\mu - t|^{q/(q-1)} dt \right)^{1-1/q} \left[ \int_0^1 \left( t^s \left| f'\left(\frac{a+b}{2}\right) \right|^q + (1-t)^s |f'(b)|^q \right) dt \right]^{1/q} \right\}. \end{aligned}$$

If  $q = 1$ , we have

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} \left[ \int_0^1 |1 - \lambda - t| \left| f'\left(ta + (1-t)\frac{a+b}{2}\right) \right| dt + \int_0^1 |\mu - t| \left| f'\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \\ & \leq \frac{b-a}{4} \left\{ \left( \int_0^1 |1 - \lambda - t| dt \right) \left[ \int_0^1 \left( t^s |f'(a)| + (1-t)^s \left| f'\left(\frac{a+b}{2}\right) \right| \right) dt \right] \right. \\ & \quad \left. + \left( \int_0^1 |\mu - t| dt \right) \left[ \int_0^1 \left( t^s \left| f'\left(\frac{a+b}{2}\right) \right| + (1-t)^s |f'(b)| \right) dt \right] \right\}. \end{aligned}$$

Theorem 3.4 is thus proved.  $\square$

**Corollary 3.4.1.** *Under conditions of Theorem 3.4,*

(1) *when  $q = 1$  and  $\lambda = \mu$ , we have*

$$\begin{aligned} (3.10) \quad & \left| \frac{\lambda[f(a) + f(b)]}{2} + (1 - \lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4(s+1)} \left( \frac{1}{2} - \lambda + \lambda^2 \right) \left[ |f'(a)| + 2 \left| f'\left(\frac{a+b}{2}\right) \right| + |f'(b)| \right] \\ & \leq \frac{b-a}{2^{s+1}(s+1)} \left( \frac{1}{2} - \lambda + \lambda^2 \right) (2^{s-1} + 1) [|f'(a)| + |f'(b)|]; \end{aligned}$$

(2) *when  $q > 1$  and  $\lambda = \mu$ , we have*

$$\begin{aligned} & \left| \frac{\lambda[f(a) + f(b)]}{2} + (1 - \lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left( \frac{1}{s+1} \right)^{1/q} \left[ \lambda^{(2q-1)/(q-1)} + (1-\lambda)^{(2q-1)/(q-1)} \right]^{1-1/q} \\ & \quad \times \left\{ \left[ |f'(a)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + \left[ \left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(b)|^q \right]^{1/q} \right\} \\ & \leq \frac{b-a}{2^{s/q+2}} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left( \frac{1}{s+1} \right)^{1/q} \left[ \lambda^{(2q-1)/(q-1)} + (1-\lambda)^{(2q-1)/(q-1)} \right]^{1-1/q} \end{aligned}$$

$$\times \{ [(2^s + 1)|f'(a)|^q + |f'(b)|^q]^{1/q} + [|f'(a)|^q + (2^s + 1)|f'(b)|^q]^{1/q} \}.$$

**Corollary 3.4.2.** *Under conditions of Theorem 3.4,*

(1) *when  $q = 1$  and  $s = 1$ , we have*

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4(s+1)} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right) \left[ |f'(a)| + \left| f'\left(\frac{a+b}{2}\right) \right| \right] + \left( \frac{1}{2} - \mu + \mu^2 \right) \left[ \left| f'\left(\frac{a+b}{2}\right) \right| + |f'(b)| \right] \right\} \\ & \leq \frac{b-a}{16} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right) [3|f'(a)| + |f'(b)|] + \left( \frac{1}{2} - \mu + \mu^2 \right) [|f'(a)| + 3|f'(b)|] \right\}; \end{aligned}$$

(2) *when  $q > 1$  and  $s = 1$ , we have*

$$\begin{aligned} & \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2^{1/q+2}} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left\{ [\lambda^{(2q-1)/(q-1)} + (1-\lambda)^{(2q-1)/(q-1)}]^{1-1/q} \left[ \left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(a)|^q \right]^{1/q} \right. \\ & \quad \left. + [\mu^{(2q-1)/(q-1)} + (1-\mu)^{(2q-1)/(q-1)}]^{1-1/q} \left[ \left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(b)|^q \right]^{1/q} \right\} \\ & \leq \frac{b-a}{2^{2/q+2}} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left\{ [\lambda^{(2q-1)/(q-1)} + (1-\lambda)^{(2q-1)/(q-1)}]^{1-1/q} [3|f'(a)|^q + |f'(b)|^q]^{1/q} \right. \\ & \quad \left. + [\mu^{(2q-1)/(q-1)} + (1-\mu)^{(2q-1)/(q-1)}]^{1-1/q} [|f'(a)|^q + 3|f'(b)|^q]^{1/q} \right\}. \end{aligned}$$

#### 4. APPLICATIONS TO MEANS

Finally, we apply some inequalities of Hermite-Hadamard type for extended  $s$ -convex functions to construct some inequalities for means.

For two positive numbers  $a > 0$  and  $b > 0$ , let

$$(4.1) \quad A(a, b) = \frac{a+b}{2} \quad \text{and} \quad L_s(a, b) = \begin{cases} \left[ \frac{b^{s+1} - a^{s+1}}{(s+1)(b-a)} \right]^{1/s}, & s \neq 0, -1 \text{ and } a \neq b, \\ \frac{\ln b - \ln a}{b-a}, & s = -1 \text{ and } a \neq b, \\ \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{1/(b-a)}, & s = 0 \text{ and } a \neq b, \\ a, & a = b. \end{cases}$$

They are called the arithmetic and generalized logarithmic means of two positive numbers  $a$  and  $b$  respectively.

Let  $f(x) = x^s$  for  $x > 0$ ,  $s > 0$ , and  $q \geq 1$ . If  $0 \leq (s-1)q \leq 1$  and  $0 \leq s-1 \leq 1$ , then

$$\begin{aligned} |f'(tx + (1-t)y)|^q & \leq s^q [t^{(s-1)q} x^{(s-1)q} + (1-t)^{(s-1)q} y^{(s-1)q}] \\ & \leq t^{s-1} |f'(x)|^q + (1-t)^{s-1} |f'(y)|^q \end{aligned}$$

for  $x, y > 0$  and  $t \in (0, 1)$ . If  $-1 < (s-1)q \leq 0$  and  $-1 < s-1 \leq 0$ , then

$$|f'(tx + (1-t)y)|^q \leq t^{s-1} |f'(x)|^q + (1-t)^{s-1} |f'(y)|^q$$

for  $x, y > 0$  and  $t \in (0, 1)$ . If  $-1 < (s-1)q \leq 1$  and  $-1 < s-1 \leq 1$ , then  $|f'(x)|^q = |s|^q x^{(s-1)q}$  is an extended  $(s-1)$ -convex function on  $[a, b]$ .

Applying Corollary 3.1.2 to  $|s|^q x^{(s-1)q}$  yields the following theorem.

**Theorem 4.1.** *Let  $b > a > 0$ ,  $q \geq 1$ ,  $0 < s \leq 2$ ,  $-1 < (s-1)q \leq 1$ , and  $0 \leq \lambda \leq 1$ . Then*

$$(4.2) \quad \begin{aligned} |\lambda A(a^s, b^s) + (1-\lambda)A^s(a, b) - L_s^s(a, b)| &\leq \frac{(b-a)s}{2^{(s-1)/q+2}} \left[ \frac{1}{s(s+1)} \right]^{1/q} \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} \\ &\quad \times \left\{ \left[ (2(2-\lambda)^{s+1} + 2^s((s+1)\lambda - 2) + (s+1)\lambda - s - 2)a^{(s-1)q} \right. \right. \\ &\quad \left. \left. + (2\lambda^{s+1} + s - (s+1)\lambda)b^{(s-1)q} \right]^{1/q} + \left[ (2\lambda^{s+1} + s - (s+1)\lambda)a^{(s-1)q} \right. \right. \\ &\quad \left. \left. + (2(2-\lambda)^{s+1} + 2^s((s+1)\lambda - 2) + (s+1)\lambda - s - 2)b^{(s-1)q} \right]^{1/q} \right\}. \end{aligned}$$

Specially, if  $q = 1$ , then

$$(4.3) \quad \begin{aligned} |\lambda A(a^s, b^s) + (1-\lambda)A^s(a, b) - L_s^s(a, b)| \\ \leq \frac{(b-a)s}{2^{s-1}s(s+1)} \{ (2-\lambda)^{s+1} + \lambda^{s+1} + [(s+1)\lambda - 2]2^{s-1} - 1 \} A(a^{s-1}, b^{s-1}). \end{aligned}$$

Taking  $f(x) = x^s$  for  $x > 0$  and  $s > 0$  in Corollary 3.2.2 derives the following inequalities for means.

**Theorem 4.2.** *Let  $b > a > 0$ ,  $q \geq 1$ ,  $0 < s \leq 2$ ,  $-1 < (s-1)q \leq 1$ , and  $0 \leq \lambda \leq 1$ . Then*

$$\begin{aligned} |\lambda A(a^s, b^s) + (1-\lambda)A^s(a, b) - L_s^s(a, b)| &\leq \frac{(b-a)s}{4} \left[ \frac{1}{s(s+1)} \right]^{1/q} \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} \\ &\quad \times \left\{ \left[ (2(1-\lambda)^{s+1} + (s+1)\lambda - 1)a^{(s-1)q} + (2\lambda^{s+2} - (s+1)\lambda + s)A^{(s-1)q}(a, b) \right]^{1/q} \right. \\ &\quad \left. + \left[ (2\lambda^{s+1} - (s+1)\lambda + s)A^{(s-1)q}(a, b) + (2(1-\lambda)^{s+1} + (s+1)\lambda - 1)b^{(s-1)q} \right]^{1/q} \right\}. \end{aligned}$$

In particular, if  $q = 1$ , then

$$\begin{aligned} |\lambda A(a^s, b^s) + (1-\lambda)A^s(a, b) - L_s^s(a, b)| \\ \leq \frac{(b-a)s}{2s(s+1)} \{ [2(1-\lambda)^{s+1} + (s+1)\lambda - 1]A(a^{s-1}, b^{s-1}) + [2\lambda^{s+1} + s - (s+1)\lambda]A^{s-1}(a, b) \}. \end{aligned}$$

Letting  $f(x) = x^s$  for  $x > 0$  and  $s > 0$  in Corollary 3.3.1 generates inequalities below.

**Theorem 4.3.** *Let  $b > a > 0$ ,  $q \geq 1$ ,  $0 < s \leq 2$ , and  $0 \leq \lambda \leq 1$ .*

(1) *If  $q > 1$ , then*

$$\begin{aligned} |\lambda A(a^s, b^s) + (1-\lambda)A^s(a, b) - L_s^s(a, b)| &\leq \frac{(b-a)s}{2^{s/q+2}} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left( \frac{1}{s+1} \right)^{1/q} [\lambda^{(2q-1)/(q-1)} \\ &\quad + (1-\lambda)^{(2q-1)/(q-1)}]^{1-1/q} \{ [(2^s-1)a^{(s-1)q} + b^{(s-1)q}]^{1/q} + [a^{(s-1)q} + (2^s-1)b^{(s-1)q}]^{1/q} \}. \end{aligned}$$

(2) *If  $q = 1$ , then*

$$(4.4) \quad \left| \lambda A(a^s, b^s) + (1-\lambda)A^s(a, b) - L_s^s(a, b) \right| \leq \frac{(b-a)s}{s+1} \left( \frac{1}{2} - \lambda + \lambda^2 \right) A(a^{s-1}, b^{s-1}).$$

From Corollary 3.4.1, it follows that

**Theorem 4.4.** *Let  $b > a > 0$ ,  $q \geq 1$ ,  $0 < s \leq 2$ , and  $0 \leq \lambda \leq 1$ .*

(1) *If  $q > 1$  and  $-1 < (s-1)q \leq 1$ , then*

$$\left| \lambda A(a^s, b^s) + (1-\lambda)A^s(a, b) - L_s^s(a, b) \right| \leq \frac{(b-a)s}{4} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left( \frac{1}{s+1} \right)^{1/q} [\lambda^{(2q-1)/(q-1)} + (1-\lambda)^{(2q-1)/(q-1)}]^{1-1/q} \{ [a^{(s-1)q} + A^{(s-1)q}(a, b)]^{1/q} + [A^{(s-1)q}(a, b) + b^{(s-1)q}]^{1/q} \}.$$

(2) *If  $q = 1$ , then*

$$(4.5) \quad \left| \lambda A(a^s, b^s) + (1-\lambda)A^s(a, b) - L_s^s(a, b) \right| \leq \frac{(b-a)s}{2(s+1)} \left( \frac{1}{2} - \lambda + \lambda^2 \right) [A(a^{s-1}, b^{s-1}) + A^{s-1}(a, b)].$$

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(B.-Y. Xi) COLLEGE OF MATHEMATICS, INNER MONGOLIA UNIVERSITY FOR NATIONALITIES, TONGLIAO CITY, INNER MONGOLIA AUTONOMOUS REGION, 028043, CHINA

*E-mail address:* baoyintu78@qq.com, baoyintu68@sohu.com

(F. Qi) DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN CITY, 300387, CHINA

*E-mail address:* qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

*URL:* <http://qifeng618.wordpress.com>